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Commutation and anti-commutation relations for a class of Gel'fand-Yaglom matrices

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Abstract. A realization is given, in terms of Dirac matrices and suitable SL(2, c) generators, of Gel'fand–Yaglom matrices associated with the simplest parity-invariant wave equations for particles with half-odd-integral spin. Commutation and anti-commutation relations for these matrices are thereby derived, replacing those given previously by Lorente, Huddleston and Roman, which are shown to be incorrect.

1. Introduction and summary

Relativistic wave equations of the form:

$$\begin{aligned} &(L_\mu p^\mu - \kappa)\psi(x) = 0 \\ &p^\mu = \mathrm{i}\partial/\partial x_\mu, \qquad \mu = 0, 1, 2, 3 \end{aligned} \tag{1}$$

have been widely studied in varying degrees of generality, notably by Gel'fand and Yaglom (1948). Here ψ denotes an indexed column of functions and the index space carries a representation of SL(2, c), either finite- or infinite-dimensional. The generators $J_{\mu\nu}(= -J_{\nu\mu})$ of this representation are matrices (or operators) acting in the index space and satisfying

$$\mathrm{i}[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\nu\rho}J_{\mu\sigma} - g_{\mu\sigma}J_{\nu\rho}. \tag{2}$$

(The metric tensor is diagonal, with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$.) The matrices L_μ and κ also act in the index space and satisfy

$$\mathrm{i}[L_\mu, J_{\nu\rho}] = g_{\mu\rho}L_\nu - g_{\mu\nu}L_\rho \tag{3}$$

$$[\kappa, J_{\mu\nu}] = 0 \tag{4}$$

in order to secure the relativistic invariance of the wave equation.

In the cases usually discussed, κ is a scalar multiple of the unit matrix, whence equation (4) is trivially satisfied. Furthermore, the representation of SL(2, c) is assumed to be reducible to the direct sum of a finite number of irreducible representations. It is known that only for *some* representations of this type do there exist non-trivial L_μ satisfying equation (3), and in every such case the matrix elements of L_μ have been determined in a particular basis (Gel'fand and Yaglom 1948, Gel'fand *et al* 1963).

However, these results cannot be regarded as providing a completely satisfactory classification of wave equations of the type indicated, as Wightman (1968) has pointed out. For example, Dirac's equation for the electron is of this form, but the structure of the equation in this case is not well revealed by writing down an explicit representation of the Dirac matrices. It is of greater value in that case to realize that the matrices can be characterized completely, and *in a representation-independent way*, by specifying the algebraic relations they satisfy, in particular (writing $L_\mu = \gamma_\mu$ in this case):

$$[\gamma_\mu, \gamma_\nu] = -4\mathrm{i}J_{\mu\nu} \tag{5}$$

and

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{6}$$

Therefore, in view of the potential importance of Gel'fand–Yaglom equations in the description of particles, there is some motivation for further study of the algebraic properties of the associated matrices L_μ. On the other hand there seems to be little chance of classifying *all* equations of this type in terms of such properties. One could propose algebraic relations for the L_μ, perhaps generalizing equations (5), (6) in some way, and thence deduce the form of the corresponding representation of SL(2, c), the masses and spins of the particles described by the corresponding equation (1), and so on. In this connection one recalls the work of Lubanski (1942), Bhabha (1945) and Harish-Chandra (1948) in particular.

An alternative approach is to select from the set of all Gel'fand–Yaglom equations a subset of equations which might be expected to have particularly simple structures, and to attempt to find characteristic algebraic properties of the corresponding matrices L_μ. From this point of view there is a great variety of possible subsets to consider, and one must therefore select only types likely to be of special interest.

In this paper we shall consider what are, as has previously been indicated (Feldman and Mathews 1966, Stoyanov and Todorov 1968), the simplest parity-invariant equations for particles with half-odd-integral spin. For a typical such equation the wavefunctions ψ belong to a representation of SL(2, c) labelled

$$C(\alpha) \equiv (\tfrac{1}{2}, \alpha) \oplus (-\tfrac{1}{2}, \alpha),$$

where α is some corresponding complex number. (Irreducible representations are labelled (k_0, c) where $2k_0$ is integral and c is complex. The invariants of SL(2, c) are $\frac{1}{2}J_{\mu\nu}J^{\mu\nu}$ and

$$Q = -\tfrac{1}{2}\mathrm{i}\tilde{J}_{\mu\nu}J^{\mu\nu} \tag{7}$$

where $\tilde{J}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}$ with $\epsilon_{0123} = -1$. On the representation (k_0, c), $\frac{1}{2}J_{\mu\nu}J^{\mu\nu}$ and Q take the values $(k_0^2 + c^2 - 1)$ and $2k_0c$ respectively.) Then for given α,

$$\tfrac{1}{2}J_{\mu\nu}J^{\mu\nu} = \alpha^2 - \tfrac{3}{4}, \tag{8}$$

while Q takes the value α on $(\frac{1}{2}, \alpha)$ and $-\alpha$ on $(-\frac{1}{2}, \alpha)$, so that

$$Q^2 = \alpha^2. \tag{9}$$

If $\alpha - \frac{1}{2}$ is integral and $|\alpha| \geqslant \frac{3}{2}$, then $C(\alpha)$ is finite-dimensional; if α is pure imaginary, then $C(\alpha)$ is infinite-dimensional and unitary; and for all other values of α, $C(\alpha)$ is infinite-dimensional and non-unitary. The representations $C(\alpha)$ and $C(-\alpha)$ are equivalent. It is evident from the results of Gel'fand and Yaglom (1948) that for each value of α there are non-trivial L_μ satisfying equation (3). These matrices 'link' the representations $(\frac{1}{2}, \alpha)$ and $(-\frac{1}{2}, \alpha)$, as does the parity transformation. Indeed, once the generators $J_{\mu\nu}$ of $C(\alpha)$ are completely specified and the precise form of the parity transformation between $(\frac{1}{2}, \alpha)$ and $(-\frac{1}{2}, \alpha)$ is defined, the requirement of parity invariance of the wave equation (1) completely specifies the L_μ, up to multiplication by an arbitrary complex scalar. This is clearly illustrated in the case $\alpha = \frac{3}{2}$, corresponding to Dirac's equation, which is typical in this respect. The arbitrary complex scalar multiplying L_μ may be chosen in any way which is convenient without loss of generality as regards equation (1), where κ is by assumption an arbitrary scalar multiple of the unit matrix.

Our main results follow from the presentation, for arbitrary α, of a realization of the L_μ in terms of more familiar entities—Dirac matrices and suitable SL(2, c) generators. This realization is given in § 3. Using it in § 4, we are able to calculate in particular the commutator and anti-commutator of L_μ and L_ν, obtaining (for a suitable choice of the scalar multiplying L_μ)

$$[L_\mu, L_\nu] = -\mathrm{i}J_{\mu\nu} - 2\tilde{J}_{\mu\nu}Q \tag{10}$$

and

$$\{L_\mu, L_\nu\} = 2g_{\mu\nu}(\alpha^2 - \tfrac{1}{2}) - \{J_{\mu\sigma}, J_\nu{}^\sigma\}. \tag{11}$$

These relations generalize those given above for Dirac matrices which, as has already been mentioned, now appear in the particular case $\alpha = \frac{3}{2}$. There the relations (5), (6) imply that $Q = 3/2\gamma_5 (= 3/2\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3)$, that $\tilde{J}_{\mu\nu} = \mathrm{i}J_{\mu\nu}\gamma_5$ and that $\{J_{\mu\sigma}, J_\nu{}^\sigma\} = \frac{3}{2}g_{\mu\nu}$, so that relations (10) and (11) reduce to (5) and (6) respectively.

The infinite-dimensional case $\alpha = 0$ is also of special interest, as the L_μ are then the (half-odd-integer spin) Majorana matrices. There $Q = 0$ and (10) reduces to the well known commutation relation for these matrices. The result (11), with $\alpha = 0$, agrees with that given for the Majorana matrices by Böhm (1968).

In the general case some information can be obtained from equations (8) and (11) about the mass and spin spectra of the corresponding wave equation. We have (Bracken 1970)

$$\begin{aligned}(L_\mu p^\mu)^2 &= \tfrac{1}{2}\{L_\mu, L_\nu\}p^\mu p^\nu \\ &= (\alpha^2 - \tfrac{1}{2})p_\mu p^\mu - \tfrac{1}{2}\{J_{\mu\sigma}, J_\nu{}^\sigma\}p^\mu p^\nu \\ &= \tfrac{1}{4}p_\mu p^\mu - \omega_\mu\omega^\mu\end{aligned} \tag{12}$$

where $\omega_\mu = \tilde{J}_{\mu\nu}p^\nu$ is the Pauli–Lubanski vector, so that

$$\omega_\mu\omega^\mu \equiv \tfrac{1}{2}\{J_{\mu\sigma}, J_\nu{}^\sigma\}p^\mu p^\nu - \tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}p_\sigma p^\sigma.$$

Hence equation (1) implies that

$$(\tfrac{1}{4}p_\mu p^\mu - \omega_\mu\omega^\mu)\psi = \kappa^2\psi. \tag{13}$$

If ψ is a wavefunction for a particle with mass m and spin s, then in addition to (1) and (13),

$$\begin{aligned}p_\mu p^\mu\psi &= m^2\psi \\ \omega_\mu\omega^\mu\psi &= -s(s+1)m^2\psi\end{aligned} \tag{14}$$

so that

$$(s+\tfrac{1}{2})^2 m^2\psi = \kappa^2\psi.$$

The possible values of s in equation (14) are determined by the SU(2) content of the representation $C(\alpha)$ and we can conclude that, if we take κ to be real and positive in the wave equation, it describes in particular particles with the following masses and spins:

(*a*) *finite-dimensional cases*: $\alpha-\frac{1}{2}$ *integral*, $|\alpha| \geqslant \frac{3}{2}$.

$$m = \frac{\kappa}{(s+\frac{1}{2})}, \qquad s = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, |\alpha|-1.$$

(*b*) *infinite-dimensional cases*

$$m = \frac{\kappa}{(s+\frac{1}{2})}, \qquad s = \tfrac{1}{2}, \tfrac{3}{2}, \ldots.$$

The latter result is well known in the case of the Majorana equation. In *all* the infinite-dimensional cases it can be expected that there are also space-like and light-like solutions of the wave equation (Bargmann 1949, Rühl 1967, Grodsky and Streater 1968). In any case the result (13) relates the two invariants of the Poincaré group and so restricts the representations of that group which can appear.

The results obtained here differ significantly from those presented by Lorente *et al* (1973) (to be referred to as LHR), who also have considered the class of matrices L_μ under discussion here, as well as those appearing in parity-invariant equations (1) when the representation of SL(2, C) involved is

$$K(\alpha) \equiv (\alpha, \tfrac{1}{2}) \oplus (\alpha, -\tfrac{1}{2})$$

with 2α integral. In the next section we show directly that the results of LHR are incorrect.

Fronsdal and White (1967) also have considered the equations associated with the representations $K(\alpha)$ and have obtained (in disagreement with LHR) a mass and spin relationship of the general form $m = \kappa/(s+\frac{1}{2})$. This leads us to believe that the results (10), (11) will be found to hold also in these cases (rather than those presented by LHR), but our method does not enable us to verify that here.

The commutation relation (10) suggests that, except in the Dirac and Majorana cases, the matrices L_μ and $J_{\nu\rho}$ do not belong to a Lie algebra. Nevertheless we feel that the relations (10) and (11) are sufficiently simple to be of interest.

2. The results of Lorente, Huddleston and Roman

According to LHR the following equations are satisfied by the L_μ for arbitrary α:

(i) $[L_\mu, L_\nu] = -\mathrm{i}J_{\mu\nu}$

(ii) $[L_\mu, L_\nu L^\nu] = 0$

(iii) $\{L_\mu, L_\nu\} = \frac{4}{3}(\alpha^2-\frac{3}{4})g_{\mu\nu} - \{J_{\mu\sigma}, J_\nu{}^\sigma\}$.

The results (i) and (iii) are in conflict with relations (10), (11), but the deduction of (ii) and (iii) by LHR is based on (i), which can easily be shown to be incorrect, except in the Dirac and Majorana cases. While (iii) is also correct in those special cases, it can be concluded that it is not so in general; and while (ii) is in general correct, the proof given by LHR, being based on (i), is in general false.

In order to arrive at the equation (i) LHR argue that, since in $C(\alpha)$ no irreducible representation of SL(2, c) appears more than once, 'the Lie algebra of the $J^{\mu\sigma}$ and L^μ must close and becomes exactly that of Sp(4, R)'. This argument is simply wrong. In order to see an inconsistency it is sufficient to consider the finite-dimensional cases, when the representation of SL(2, c) is in each case $C(j+\frac{1}{2})$ for some positive integer j. If the LHR argument, and consequently the result (i), were correct, there would have to be for each j a representation of Sp(4, R) whose SL(2, c) content is just $C(j+\frac{1}{2})$. However, the reduction of finite-dimensional representations of Sp(4, R) (or SO(5)) with respect to SL(2, c) (or SO(4)) is well known (Murnaghan 1938, Lubanski 1942, Bhabha 1945, Corson 1953) and shows that there are *no* such representations, except in the case $j = 1$ (the Dirac representation). An irreducible finite-dimensional representation of Sp(4, R) can be labelled $[p, q]$, where $2p, 2q$ and $p-q$ are integers, and $p \geqslant q \geqslant 0$. The reduction with respect to SL(2, c) is

$$[p, q] \rightarrow \sum_{k_0=-q}^{q} \sum_{c=q+1}^{p+1} \oplus(k_0, c)$$

where the summations are restricted to integer values of $q-k_0$ and $p+1-c$. Only in the case $p = q = \frac{1}{2}$ with $\alpha = \frac{3}{2}$ ($j = 1$) is the representation of SL(2, c) of the general form $(\frac{1}{2}, \alpha)\oplus(-\frac{1}{2}, \alpha)$.

In a similar way it is seen that the LHR result (i) (and hence at least the derivations of (ii) and (iii)) is invalid also in the case of the representation $K(\alpha)$.

We have mentioned that the result (ii) is correct, although incorrectly derived by LHR. In fact (ii) follows at once from our result (11) which, taken with (8), implies that $L_\mu L^\mu$ is a multiple by $2\alpha^2-\frac{1}{2}$ of the unit matrix.

3. A realization of the Gel'fand–Yaglom matrices

In order to prove the results (10, 11) we introduce a realization of the matrices L_μ in terms of Dirac matrices and suitable SL(2, c) generators. This construction, involving only objects more familiar than the Gel'fand–Yaglom matrices themselves, is perhaps of intrinsic interest apart from its value in enabling us to deduce the aforementioned results.

Suppose $L_{\mu\nu}$ are the generators of the representation $(0, \alpha-\frac{1}{2})$ where α is an arbitrary complex constant, and $S_{\mu\nu}$ are the generators of the Dirac representation $C(\frac{3}{2})$,

$$S_{\mu\nu} = \tfrac{1}{4}\mathrm{i}[\gamma_\mu, \gamma_\nu]. \tag{15}$$

It is not hard to see that (Harish-Chandra 1947)

$$(0, \alpha-\tfrac{1}{2}) \otimes C(\tfrac{3}{2}) = C(\alpha) \oplus C(\alpha-1), \tag{16}$$

so that $J_{\mu\nu} = L_{\mu\nu}+S_{\mu\nu}$ are generators of the representation $C(\alpha) \oplus C(\alpha-1)$. These two representations are distinguished by the corresponding eigenvalues

$$\alpha^2-\tfrac{3}{4}, \qquad \alpha^2-2\alpha+\tfrac{1}{4},$$

of $\frac{1}{2}J_{\mu\nu}J^{\mu\nu}$. Clearly it must be possible to construct from the $L_{\mu\nu}$ and the γ_ρ a *reducible* realization of the Gel'fand–Yaglom matrices corresponding to the representation of SL(2, c) generated by $J_{\mu\nu}$, and to isolate irreducible realizations by fixing the value of $\frac{1}{2}J_{\mu\nu}J^{\mu\nu}$.

We introduce the operator Q as in (7) and note that in this case

$$Q = \gamma_5(R+\tfrac{1}{2}) \tag{17}$$

where

$$R = L_{\mu\nu}S^{\mu\nu}+1 \tag{18}$$

$$\begin{aligned} R^2 &= \tfrac{1}{2}L_{\mu\nu}L^{\mu\nu}+1 \\ &= (\alpha-\tfrac{1}{2})^2 \end{aligned} \tag{19}$$

and

$$(R+\tfrac{1}{2})^2 = Q^2 = \tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}+\tfrac{3}{4}. \tag{20}$$

Define

$$L_\mu = \tfrac{1}{2}\{\gamma_\mu, R\} \tag{21}$$

and note that since by virtue of equation (19) $[\gamma_\mu, R^2] = 0$, one has

$$[L_\mu, R] = 0. \tag{22}$$

Since it is also clear that $\{L_\mu, \gamma_5\} = 0$, one has

$$\{L_\mu, Q\} = 0 \tag{23}$$

and

$$[L_\mu, \tfrac{1}{2}J_{\nu\rho}J^{\nu\rho}] = 0. \tag{24}$$

It follows from the last two equations that the L_μ leave separately invariant the subspaces associated with $C(\alpha)$ and $C(\alpha-1)$, but link the irreducible representations of $SL(2, C)$ within each such subspace. They are therefore (up to multiplication by an arbitrary constant) the required reducible realization of the Gel'fand–Yaglom matrices. (Note that in the case $L_{\mu\nu} = 0$ ($R = 1, \alpha = \frac{3}{2}$), L_μ reduces trivially to γ_μ.) We shall work with this reducible realization to deduce the results (10), (11). The results for the irreducible matrices corresponding to the representation $C(\alpha)$ of $SL(2, C)$ are obtained by setting $\frac{1}{2}J_{\mu\nu}J^{\mu\nu} = \alpha^2-\frac{3}{4}$, or equivalently, in view of equations (19) and (20), by setting $R = \alpha-\frac{1}{2}$.

4. The commutation and anti-commutation relations

We begin by proving the result (11) (Bracken 1970). Note firstly that

$$L_\mu = -iD_\mu+\gamma_\mu R \tag{25}$$

where

$$D_\mu = -\tfrac{1}{2}i[\gamma_\mu, R] = L_{\mu\nu}\gamma^\nu, \tag{26}$$

and that, in consequence,

$$\{D_\mu, R\} = 0. \tag{27}$$

Then

$$\{L_\mu, L_\nu\} = -\{D_\mu, D_\nu\}-i\{D_\mu, \gamma_\nu R\}-i\{D_\nu, \gamma_\mu R\}+\{\gamma_\mu R, \gamma_\nu R\}. \tag{28}$$

It is readily checked that

$$D_\mu D_\nu = -L_{\mu\sigma}L^\sigma{}_\nu-i(L_{\mu\rho}L_{\nu\sigma}-L_{\mu\sigma}L_{\nu\rho})S^{\rho\sigma} \tag{29}$$

so that

$$-\{D_\mu, D_\nu\} = L_{\mu\sigma}L^\sigma{}_\nu+L_{\nu\sigma}L^\sigma{}_\mu+2g_{\mu\nu}(R-1)+2(L_{\mu\sigma}S^\sigma{}_\nu+L_{\nu\sigma}S^\sigma{}_\mu). \tag{30}$$

Furthermore, using equation (27) we find

$$\begin{aligned}\{D_\mu, \gamma_\nu R\} &= [D_\mu, \gamma_\nu]R \\ &= -4iL_{\mu\sigma}S^\sigma{}_\nu R\end{aligned} \tag{31}$$

so that

$$-\mathrm{i}\{D_\mu, \gamma_\nu R\} - \mathrm{i}\{D_\nu, \gamma_\mu R\} = -4(L_{\mu\sigma}S^\sigma{}_\nu + L_{\nu\sigma}S^\sigma{}_\mu)R. \tag{32}$$

Using equation (26) we find

$$\begin{aligned}\{\gamma_\mu R, \gamma_\nu R\} &= 2g_{\mu\nu}R^2 - 2\mathrm{i}(\gamma_\mu D_\nu + \gamma_\nu D_\mu)R \\ &= 2g_{\mu\nu}R^2 + 4(L_{\mu\sigma}S^\sigma{}_\nu + L_{\nu\sigma}S^\sigma{}_\mu)R.\end{aligned} \tag{33}$$

Combining equations (28), (30), (32) and (33) and noting that

$$\begin{aligned}\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu} &= \tfrac{3}{2} \\ S_{\mu\sigma}S^\sigma{}_\nu &= -\tfrac{1}{2}\gamma_\mu\gamma_\nu - \tfrac{1}{4}g_{\mu\nu},\end{aligned} \tag{34}$$

we obtain

$$\begin{aligned}\{L_\mu, L_\nu\} &= L_{\mu\sigma}L^\sigma{}_\nu + L_{\nu\sigma}L^\sigma{}_\mu + S_{\mu\sigma}S^\sigma{}_\nu + S_{\nu\sigma}S^\sigma{}_\mu + 2(L_{\mu\sigma}S^\sigma{}_\nu + L_{\nu\sigma}S^\sigma{}_\mu) + 2g_{\mu\nu}(R^2 + R - \tfrac{1}{4}), \\ &= 2g_{\mu\nu}(\tfrac{1}{2}J_{\rho\sigma}J^{\rho\sigma} + \tfrac{1}{4}) - \{J_{\mu\sigma}, J_\nu{}^\sigma\}\end{aligned}$$

which reduces to (11) as required, when $\frac{1}{2}J_{\rho\sigma}J^{\rho\sigma} = \alpha^2 - \frac{3}{4}$.

In order to prove the result (10) we begin by noting that

$$\begin{aligned}\{R, S_{\mu\nu}\} &= [R, S_{\mu\nu}] + 2S_{\mu\nu}R \\ &= -2\mathrm{i}(L_{\mu\sigma}S^\sigma{}_\nu - L_{\nu\sigma}S^\sigma{}_\mu) + 2S_{\mu\nu}R\end{aligned}$$

which, from the definition of R and the properties of Dirac matrices, reduces to

$$\{R, S_{\mu\nu}\} = L_{\mu\nu} - \mathrm{i}\gamma_5\tilde{L}_{\mu\nu} + 2S_{\mu\nu} \tag{35}$$

where the dual tensor of $L_{\mu\nu}$ is defined by analogy with $\tilde{J}_{\mu\nu}$. Now from equation (25) we have

$$\begin{aligned}[L_\mu, L_\nu] &= -[D_\mu, D_\nu] - \mathrm{i}(D_\mu\gamma_\nu - D_\nu\gamma_\mu)R + \mathrm{i}(\gamma_\mu D_\nu - \gamma_\nu D_\mu)R + [\gamma_\mu, \gamma_\nu]R^2 \\ &\quad + \gamma_\mu[R, \gamma_\nu]R - \gamma_\nu[R, \gamma_\mu]R \\ &= -[D_\mu, D_\nu] - \mathrm{i}[D_\mu, \gamma_\nu]R + \mathrm{i}[D_\nu, \gamma_\mu]R - 4\mathrm{i}S_{\mu\nu}R^2.\end{aligned} \tag{36}$$

We note that from equation (26),

$$\begin{aligned}-[D_\mu, D_\nu] &= \tfrac{1}{2}\mathrm{i}[D_\mu, \gamma_\nu R] - \tfrac{1}{2}\mathrm{i}[D_\nu, \gamma_\mu R] \\ &= \mathrm{i}\{R, L_{\mu\nu}\}\end{aligned} \tag{37}$$

and that

$$\begin{aligned}-\mathrm{i}[D_\mu, \gamma_\nu] + \mathrm{i}[D_\nu, \gamma_\mu] &= 2\mathrm{i}[R, L_{\mu\nu}], \\ &= 2\mathrm{i}\{R, L_{\mu\nu}\} - 4\mathrm{i}L_{\mu\nu}R,\end{aligned} \tag{38}$$

so that combining equations (36)–(38) we have

$$\begin{aligned}[L_\mu, L_\nu] &= \mathrm{i}\{R, L_{\mu\nu}\}(2R+1) - 4\mathrm{i}J_{\mu\nu}R^2 \\ &= \mathrm{i}\{R, J_{\mu\nu}\}(2R+1) - 4\mathrm{i}J_{\mu\nu}R^2 - \mathrm{i}\{R, S_{\mu\nu}\}(2R+1) \\ &= 2\mathrm{i}J_{\mu\nu}R - \mathrm{i}\{R, S_{\mu\nu}\}(2R+1).\end{aligned}$$

Now using equation (35) and noting that

$$S_{\mu\nu} = -\mathrm{i}\gamma_5\tilde{S}_{\mu\nu}, \tag{39}$$

we have

$$\begin{aligned}[L_\mu, L_\nu] &= -\mathrm{i}J_{\mu\nu} - \gamma_5\tilde{J}_{\mu\nu}(2R+1) \\ &= -\mathrm{i}J_{\mu\nu} - 2\tilde{J}_{\mu\nu}Q\end{aligned}$$

as required.

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